कक्षा - 12

Chapter - 7

समाकलन

निश्चित समाकलनों के गुणधर्म

Part - 19

Lakhan Saini

निश्चित समाकलनों के गुणधर्म

- निश्चित समाकलनों के गुणधर्मों के उपयोग से इनके मान सरलता से ज्ञात किए जा सकते हैं। आइए एक-एक करके इनकी चर्चा करते है।

निश्चित समाकलनों के गुणधर्म

गुणधर्म–1 : $\int_a^b f(x)dx = \int_a^b f(t)dt$

→ चर बदल

Proof :- माना $x = t$

$dx = dt$

यदि $x = a \Rightarrow t = a$

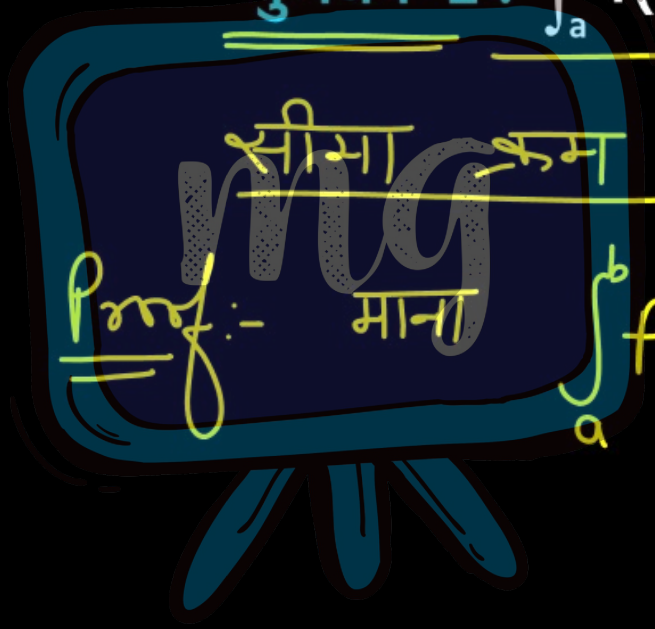
तथा $x = b \Rightarrow t = b$

$\therefore \int_a^b f(x)dx = \int_a^b f(t)dt$

निश्चित समाकलनों के गुणधर्म

गुणधर्म–2 : $\int_a^b f(x)dx = -\int_b^a f(x)dx$

सीमा क्रम बदल दें



Proof :- माना $\int_a^b f(x)dx = [F(x)]_a^b$

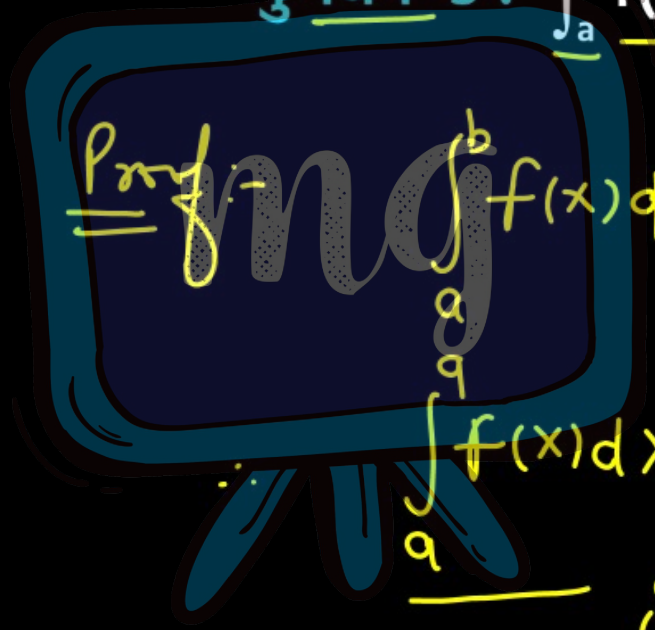
$= F(b) - F(a)$

$= -[F(a) - F(b)]$

$= -\int_b^a f(x)dx$

निश्चित समाकलनों के गुणधर्म

गुणधर्म–3 : $\int_a^a f(x)dx = 0$



Proof :-

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

$$\therefore \int_a^a f(x)dx = -\int_a^a f(x)dx$$

$$2\int_a^a f(x)dx = 0 \Rightarrow \int_a^a f(x)dx = 0$$

निश्चित समाकलनों के गुणधर्म

***गुणधर्म–4 :** Modulus मापांक

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

यहाँ a, b, c वास्तविक संख्याएं है।

$$\int_a^b f(x)dx = \int_a^{a_1} f(x)dx + \int_{a_1}^{a_2} f(x)dx + \cdots + \int_{\cdots}^{b} f(x)dx$$

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Proof:- माना $\int_a^b f(x)\,dx = [F(x)]_a^b$

$= F(b) - F(a)$

$\therefore \int_a^c f(x)\,dx + \int_c^b f(x)\,dx = [F(x)]_a^c + [F(x)]_c^b$

$= F(c) - F(a) + F(b) - F(c)$

$= F(b) - F(a)$

निश्चित समाकलनों के गुणधर्म

गुणधर्म–5 : $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

Note : $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

Proof :– $a+b-x = t$

$-dx = dt$

$dx = -dt$

जब $x = a$

$\Rightarrow t = b$

तथा जब $x = b$

$t = a$

$\therefore \int_a^b f(x)dx = \int_b^a f(a+b-t)(-dt)$

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$$= -\int_b^a f(a+b-t)\,dt$$

$$= \int_a^b f(a+b-t)\,dt$$

$$\boxed{\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx}$$

$$\int_0^a f(x) = \int_0^a f(a+0-x)\,dx$$

$$= \int_0^a f(a-x)\,dx$$

निश्चित समाकलनों के गुणधर्म

गुणधर्म–6 :

$$\int_0^{2a} f(x)dx = \int_0^{a} f(x)dx + \int_0^{a} f(2a-x)dx$$

Proof :-

$$\int_0^{2a} f(x)dx = \int_0^{a} f(x)dx + \int_a^{2a} f(x)dx$$

यहाँ $\int_a^{2a} f(x)dx = I_1$ (माना)

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माना $t = 2a - x \Rightarrow dt = -dx$

$\because$ जब $x = a \Rightarrow t = 2a - a = a$

तथा जब $x = 2a \Rightarrow t = 2a - 2a = 0$

$$\therefore I_1 = \int_a^{2a} f(x)\,dx = \int_a^0 f(2a-t)(-dt)$$

$$= \int_0^a f(2a-t)\,dt$$

$$= \int_0^a f(2a-x)\,dx$$

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$$\therefore \int_0^{2a} f(x)\,dx = \int_0^{a} f(x)\,dx + \int_0^{a} f(2a-x)\,dx$$

Case = 2 If $\boxed{f(x) = f(2a-x)}$

$$\int_0^{2a} f(x)\,dx = \int_0^{a} f(x)\,dx + \int_0^{a} f(x)\,dx$$

$$= 2\int_0^{a} f(x)\,dx$$

निश्चित समाकलनों के गुणधर्म

गुणधर्म–7 :

$y = f(x)$

$f(x) = f(-x)$

$f(-x) = -f(x)$

i) $\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$

यदि f एक समफलन है।

ii) $\int_{-a}^{a} f(x)dx = 0$

यदि f एक विषम फलन है।

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$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx$$

$$I_1 = \int_{-a}^{0} f(x)\,dx =$$

माना $x = -t$

$dx = -dt$

जब $x = -a \Rightarrow t = a$

तथा $x = 0 \Rightarrow t = 0$

$$I_1 = \int_{a}^{0} f(-t)\,(-dt)$$

$$= \int_{0}^{a} f(-t)\,dt = \int_{0}^{a} f(-x)\,dx$$

$$\int_{0}^{a} f(-x)\,dx + \int_{0}^{a} f(x)\,dx$$

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$$\int_{-a}^{a} f(x)\,dx = \int_{0}^{a} f(x)\,dx + \int_{0}^{a} f(x)\,dx$$

यदि $f(-x) = f(x)$

$$= 2\int_{0}^{a} f(x)\,dx$$

यदि $f(-x) = -f(x)$

$$\int_{0}^{2a} f(x)\,dx = 0$$

निश्चित समाकलनों के गुणधर्म

$-8+2=-6$

$x^3-x=0$

$x(x+1)(x-1)=0$

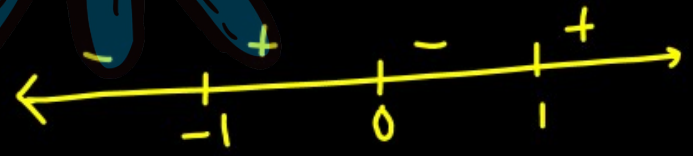
$x=-1, 0, 1$

उदाहरण-28 :

$\int_{-1}^{2} |x^3-x|\,dx$ का मान ज्ञात कीजिए।

हल :-

$$|x^3-x| = \begin{cases} -(x^3-x) & ; \quad x \le -1 \\ x^3-x & ; \quad -1 \le x < 0 \\ -(x^3-x) & ; \quad 0 \le x \le 1 \\ x^3-x & ; \quad 1 \le x \end{cases}$$



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$$\because \int_{-1}^{2} |x^3 - x|\,dx$$

$$= \int_{-1}^{0} (x^3 - x)\,dx + \int_{0}^{1} -(x^3 - x)\,dx + \int_{1}^{2} (x^3 - x)\,dx$$

$$= \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_{-1}^{0} - \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_{0}^{1} + \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_{1}^{2}$$

$$= \left[0 - \left(\frac{1}{4} - \frac{1}{2}\right)\right] - \left[\left(\frac{1}{4} - \frac{1}{2}\right) - 0\right] + \left[(4 - 2) - \left(\frac{1}{4} - \frac{1}{2}\right)\right]$$

$$= -\left(\frac{1-2}{4}\right) - \left(\frac{1-2}{4}\right) + 2 - \left(\frac{1-2}{4}\right) = 2 + \frac{3}{4} = \frac{11}{4}$$

उदाहरण-29 :

$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sin^2 x\,dx$ का मान ज्ञात कीजिए।

हल :- $\because \int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$

जब f = सम फलन है।

$\because \sin^2 x = \sin^2(-x)$

$\therefore \int_{-\pi/4}^{\pi/4} \sin^2 x\,dx = 2\int_{0}^{\pi/4} \sin^2 x\,dx$

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$$= \int_0^{\pi/4} (1 - \cos 2x)\, dx \qquad [\cos 2x = 1 - 2\sin^2 x]$$

$$= \left[x - \frac{\sin 2x}{2}\right]_0^{\pi/4}$$

$$= \left(\frac{\pi}{4} - \frac{1}{2}\right) - (0 - 0)$$

$$= \frac{\pi}{4} - \frac{1}{2}$$

उदाहरण-30 :

$\int_0^{\pi} \frac{x \sin x}{1+\cos^2 x} dx$ का मान ज्ञात कीजिए।

हल :- $\because \int_0^a f(x)dx = \int_0^a f(a-x)dx$

$\therefore \int_0^{\pi} \frac{x \sin x}{1+\cos^2 x} dx = \int_0^{\pi} \frac{(\pi-x)\sin(\pi-x)}{1+\cos^2(\pi-x)} dx$

$I = \int_0^{\pi} \frac{(\pi-x)\sin x}{1+\cos^2 x} dx$

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$$\Rightarrow I = \int_0^{\pi} \frac{\pi \sin x \, dx}{1+\cos^2 x} - \int_0^{\pi} \frac{x \sin x \, dx}{1+\cos^2 x}$$

$$I$$

$$\Rightarrow 2I = \pi \int_0^{\pi} \frac{\sin x \, dx}{1+\cos^2 x}$$

अब $\cos x = t$

$\sin x \, dx = -dt$

जब $x = 0 \Rightarrow t = 1$

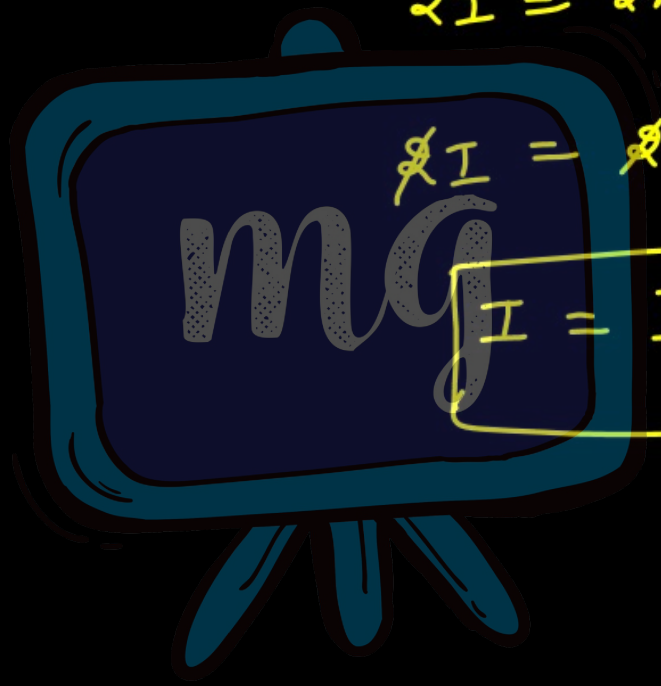
जब $x = \pi \Rightarrow t = -1$

$$\therefore 2I = \pi \int_1^{-1} \frac{-dt}{1+t^2} = \pi \int_{-1}^{1} \frac{dt}{1+t^2}$$

$$= 2\pi \int_0^{1} \frac{dt}{1+t^2}$$

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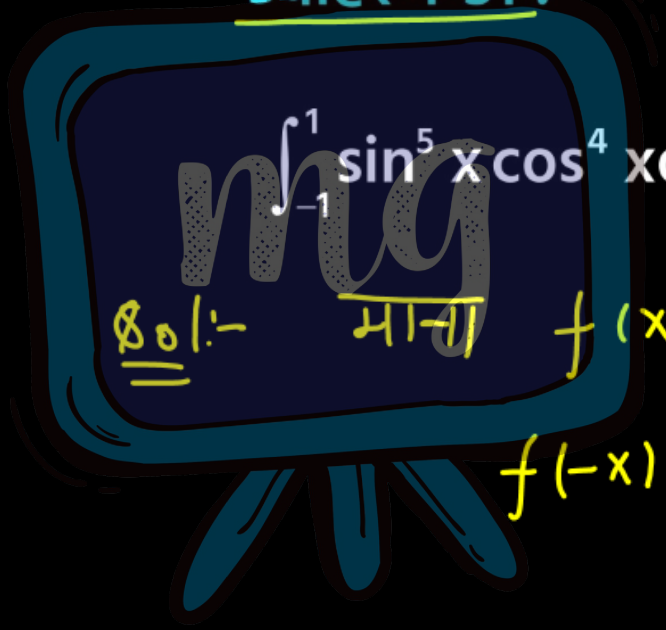
$$2I = 2\pi \left[\tan^{-1} t\right]_0^1$$

$$2I = 2\pi \left[\frac{\pi}{4} - 0\right]$$

$$\boxed{I = \frac{\pi^2}{4}}$$

उदाहरण-31 :

$\int_{-1}^{1} \sin^5 x \cos^4 x \, dx$ का मान ज्ञात कीजिए।



हल :- माना $f(x) = \sin^5 x \cos^4 x$

$f(-x) = [\sin(-x)]^5 [\cos(-x)]^4$

$= -\sin^5 x \cos^4 x$

$= -f(x)$

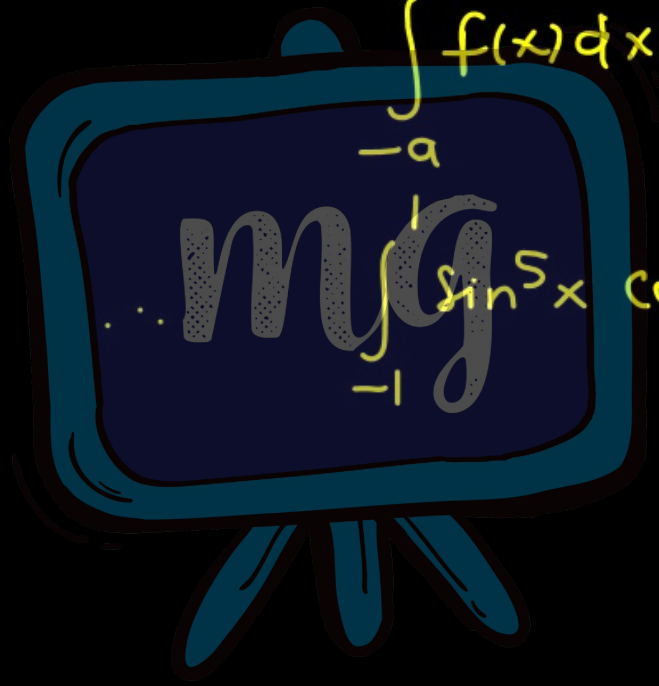
f = विषम फलन

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$$\int_{-a}^{a} f(x)\,dx = 0$$

$$\therefore \int_{-1}^{1} \sin^5 x \cos^4 x\,dx = 0$$



उदाहरण-32 :

$\int_0^{\frac{\pi}{2}} \frac{\sin^4 x}{\sin^4 x + \cos^4 x} dx$ का मान ज्ञात कीजिए।

हल :- $I = \int_0^{\pi/2} \frac{\sin^4 x}{\sin^4 x + \cos^4 x} dx$ —— (1)

$I = \int_0^{\pi/2} \frac{\left[\sin\left(\frac{\pi}{2} - x\right)\right]^4}{\left[\sin\left(\frac{\pi}{2} - x\right)\right]^4 + \left[\cos\left(\frac{\pi}{2} - x\right)\right]^4} dx$

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$$\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$$

$$I = \int_0^{\pi/2} \frac{\cos^4 x}{\cos^4 x + \sin^4 x}\,dx \quad \text{—(2)}$$

$$\therefore 2I = \int_0^{\pi/2} \frac{\cos^4 x + \sin^4 x}{\cos^4 x + \sin^4 x}\,dx$$

$$2I = \int_0^{\pi/2} dx = [x]_0^{\pi/2} = \pi/2 - 0$$

$$= \pi/2$$