कक्षा - 12

Chapter - 9

अवकल समीकरण

प्रश्नावली 9.4

(महत्वपूर्ण प्रश्न)

Part - 6

Lakhan Saini

समघातीय अवकल समीकरण

उदाहरण 12 : दर्शाइए कि अवकल समीकरण $2ye^{\frac{x}{y}}dx + (y - 2xe^{\frac{x}{y}})dy = 0$ समघातीय है और यदि, $x = 0$ जब $y = 1$ दिया हुआ हो तो इस समीकरण का विशिष्ट हल ज्ञात कीजिए।

$\frac{dy}{dx} = F(x,y)$

$F(x,y)$

समघात

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हल: दी गई अवकल समी॰ -

$$2ye^{x/y}\,dx + (y - 2xe^{x/y})\,dy = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{2ye^{x/y}}{2xe^{x/y} - y} \quad \text{—— (1)}$$

$\because$ माना $F(x,y) = \dfrac{2ye^{x/y}}{2xe^{x/y} - y}$ $\quad \left|\begin{matrix} x=1 \\ y=v \end{matrix}\right.$

$$\therefore F(\lambda x, \lambda y) = \frac{2\lambda y\, e^{\frac{\lambda x}{\lambda y}}}{2\lambda x\, e^{\frac{\lambda x}{\lambda y}} - \lambda y}$$

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$\Rightarrow F(\lambda x, \lambda y) = \lambda^0 F(x, y)$

$\therefore$ $F(x, y)$ शून्य घात का समघातीय फलन है।

$\Rightarrow$ समी॰ (i) शून्य घात वाली अवकल समी॰ है। (समघातीय)

अब, माना $y = vx$

$$\therefore \frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$\therefore v + x\frac{dv}{dx} = \frac{2ve^{\frac{x}{v}}}{2e^{\frac{x}{v}} - v}$$

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$$\Rightarrow x\frac{dv}{dx} = \frac{2ve^{v}}{2e^{v}-v} - v$$

$$\Rightarrow x\frac{dv}{dx} = \frac{2ve^{v} - 2ve^{v} + v^2}{2e^{v}-v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v^2}{2e^{v}-v}$$



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पर हल करने पर

$$\frac{2e^{v}-v}{v^2}dv = \frac{1}{x}dx$$

दोनों समाकलन करने पर

$$2\int \frac{e^{1/v}}{v^2}dv - \int \frac{1}{v}dv = \int \frac{1}{x}dx$$

यदि $e^{1/v} = t$

$$\frac{e^{1/v}dv}{v^2} = -dt \quad [v \text{ के सापेक्ष अवकलन करने पर}]$$

$$\Rightarrow -2e^{1/v} - \log v = \log x + \log c'$$

$$\Rightarrow \log v + \log x + 2e^{1/v} + \log c' = 0$$

$$\Rightarrow \log vx + 2e^{1/v} = -\log c'$$

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$\because y = vx \Rightarrow v = \frac{y}{x}$

$\log|y| + 2e^{x/y} = c \quad [c = -\log c']$ ——— (2)

जो कि समी (1) का व्यापक हल है

$\because x = 0$ है जबकि $y = 1$ हो

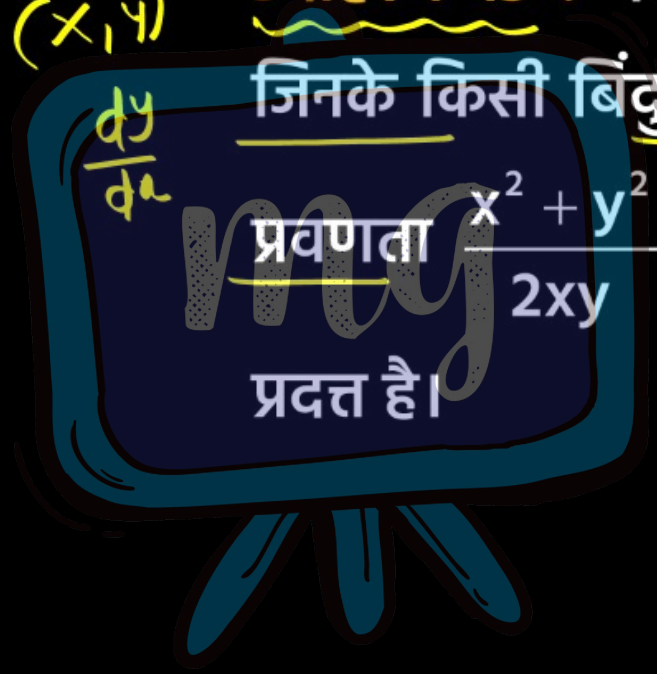
समी (2) से

$\log 1 + 2e^{0/1} = c$

$\Rightarrow c = 2$

अतः अभीष्ट विशिष्ट हल -

$$\boxed{\log|y| + 2e^{x/y} = 2}$$



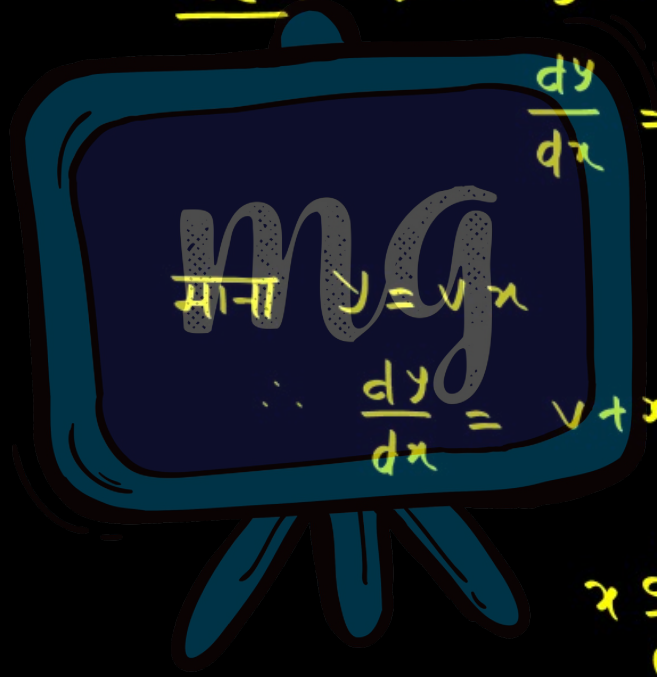
उदाहरण 13 : दर्शाइए कि वक्रों का कुल, जिनके किसी बिंदु (x, y) पर स्पर्श रेखा की प्रवणता $\frac{x^2 + y^2}{2xy}$ है, $x^2 - y^2 = cx$ द्वारा प्रदत्त है।

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हल :- प्रश्नानुसार,

$$\frac{dy}{dx} = \frac{x^2+y^2}{2xy}$$ (समघातीय अवकल समी.)



माना $y = vx$

$$\therefore \frac{dy}{dx} = v + x\frac{dv}{dx} = \frac{1+v^2}{2v}$$

$$x\frac{dv}{dx} = \frac{1+v^2}{2v} - v$$

$$x\frac{dv}{dx} = \frac{1+v^2-2v^2}{2v}$$

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$$= x\frac{dv}{dx} = \frac{1-v^2}{2v}$$

$$\Rightarrow \frac{2v\,dv}{v^2-1} = -\frac{1}{x}dx \quad [\text{चर पृथक करने पर}]$$

दोनों समाकलन करने पर -

$$\int \frac{2v\,dv}{v^2-1} = -\int \frac{1}{x}dx$$

$$\log|v^2-1| = -\log|x| + \log c^1$$

$$\log|v^2-1| + \log|x| = \log c^1$$

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$$\Rightarrow \log\left|(v^2-1)x\right| = \log c'$$

$$\Rightarrow (v^2-1)x = c'$$

$$\Rightarrow \left(\frac{y^2}{x^2}-1\right)x = c'$$

$$\Rightarrow \boxed{y^2-x^2 = cx}$$

$$\Rightarrow x^2-y^2 = -c'x$$

$$\Rightarrow x^2-y^2 = cx \quad [c = c']$$

प्रश्नावली 9.4

1. $(x^2 + xy)dy = (x^2 + y^2)dx$

$\Rightarrow \frac{dy}{dx} = \frac{x^2+y^2}{x^2+xy}$ [समघातीय अवकल समी०]

$y = vx$

$\therefore \frac{dy}{dx} = v + x\frac{dv}{dx}$

$\therefore v + x\frac{dv}{dx} = \frac{1+v^2}{1+v}$

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$$x\frac{dv}{dx} = \frac{1+v^2}{1+v} - v$$

$$\Rightarrow x\frac{dv}{dx} = \frac{1+v^2-v-v^2}{1+v}$$

$$\Rightarrow \frac{(v+1)\,dv}{v-1} = -\frac{1}{x}dx$$

$$\Rightarrow \int \frac{v-1+2}{v-1}dv = \int -\frac{1}{x}dx$$

$$= \int \left\{1+\frac{2}{v-1}\right\}dv = -\int \frac{1}{x}dx$$

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$$\Rightarrow \quad v + 2\log(v-1) = -\log x + \log c$$

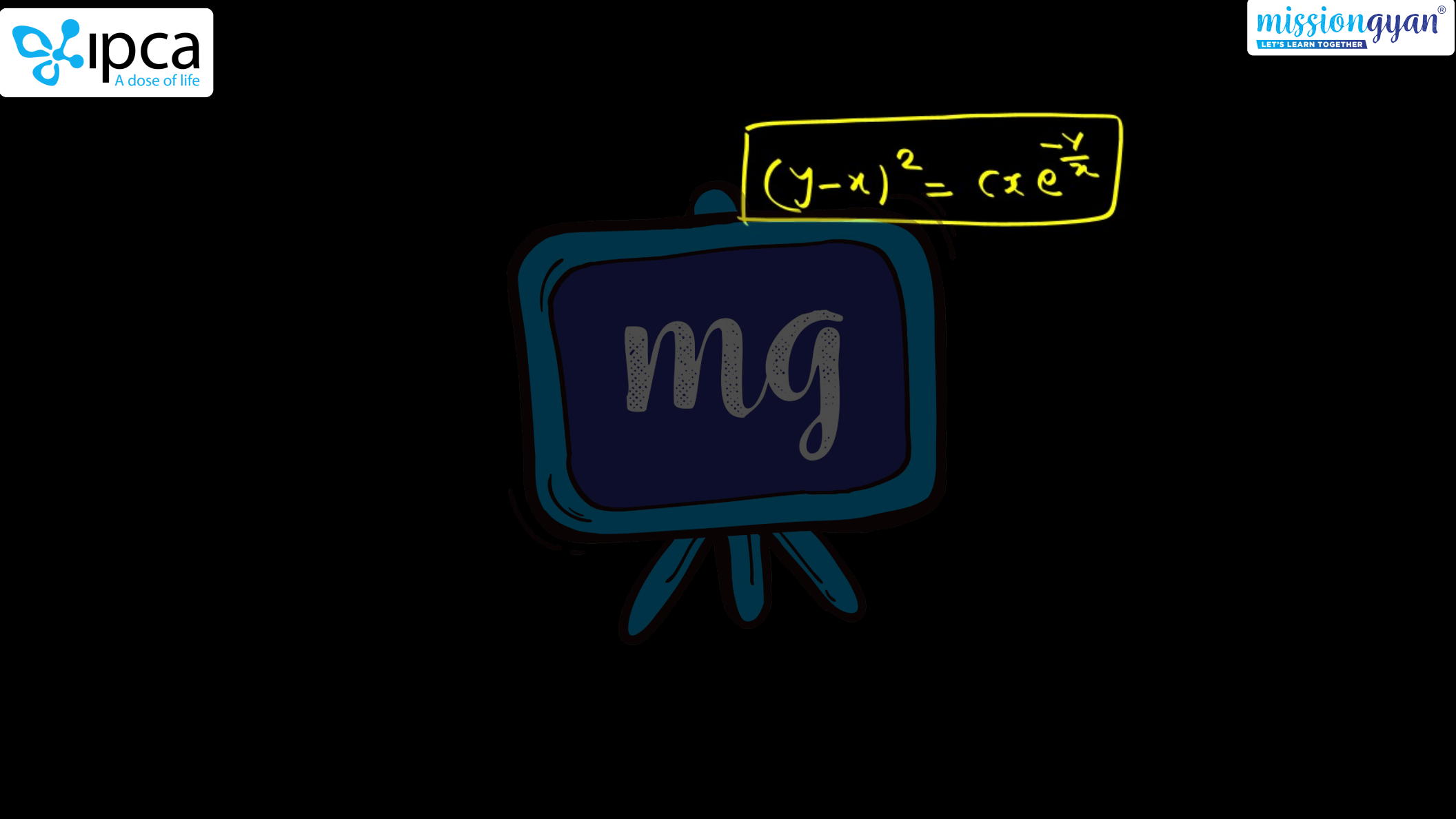
$$\Rightarrow \quad v + \log(v-1)^2 + \log x - \log c = 0$$

$$\Rightarrow \quad v + \log \frac{(v-1)^2 \cdot x}{c} = 0$$

$$\Rightarrow \quad \frac{y}{x} + \log\left[\frac{\left(\frac{y}{x}-1\right)^2 x}{c}\right] = 0$$

$$\frac{(y-x)^2}{cx} = e^{-\frac{y}{x}}$$


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$(y-x)^2 = cxe^{\frac{-y}{x}}$
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5. $x^2 \frac{dy}{dx} = x^2 - 2y^2 + xy$

$\frac{dy}{dx} = \frac{x^2 - 2y^2 + xy}{x^2}$

माना $y = vx$

$\frac{dv}{dx} = x + x\frac{dv}{dx} = \frac{1 - 2v^2 + x}{1}$

$x\frac{dv}{dx} = 1 - 2v^2$

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$$\int \frac{1}{1-2v^2}\,dv = \int \frac{1}{x}\,dx$$

$$\frac{1}{2}\int \frac{1}{\left(\frac{1}{\sqrt{2}}\right)^2 - v^2}\,dv = \int \frac{1}{x}\,dx$$

$$\left[\int \frac{1}{a^2-x^2}\,dx = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + C\right]$$

$$\Rightarrow \frac{1}{2}\cdot\frac{1}{\frac{2}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\right)}\log\left|\frac{\frac{1}{\sqrt{2}}+v}{\frac{1}{\sqrt{2}}-v}\right| = \log x + \log C$$

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$$\Rightarrow \frac{1}{2\sqrt{2}} \log \left| \frac{1+v\sqrt{2}}{1-v\sqrt{2}} \right| = \log cx$$

$v = y/x$

$$\Rightarrow \frac{1}{2\sqrt{2}} \log \left| \frac{x+y\sqrt{2}}{x-y\sqrt{2}} \right| = \log cx$$

7. $\left\{x\cos\left(\frac{y}{x}\right)+y\sin\left(\frac{y}{x}\right)\right\}ydx=\left\{y\sin\left(\frac{y}{x}\right)-x\cos\left(\frac{y}{x}\right)\right\}xdy$

$$\frac{dy}{dx}=\frac{\left\{x\cos\left(\frac{y}{x}\right)+y\sin\left(\frac{y}{x}\right)\right\}y}{\left\{y\sin\left(\frac{y}{x}\right)-x\cos\left(\frac{y}{x}\right)\right\}x}$$

(समघातीय अवकल समी०)

यदि $y=vx$

$$v+x\frac{dv}{dx}=\frac{(\cos v+v\sin v)v}{v\sin v-\cos v}$$

$$x\frac{dv}{dx}=\frac{v\cos v+v^2\sin v}{v\sin v-\cos v}-v$$

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$$x\frac{dv}{dx} = \frac{v\cos v + 2\sin v - 2\sin v + v\cos v}{v\sin v - \cos v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{2v\cos v}{v\sin v - \cos v}$$

$$\Rightarrow \int \frac{(v\sin v - \cos v)\,dv}{v\cos v} = \int \frac{2}{x}\,dx$$

$$= \int \frac{-dt}{t} = 2\int \frac{1}{x}\,dx$$

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यदि $v\cos v = t$

$$\frac{d}{dv}(v\cos v) = \frac{dt}{dv}$$

$$v(-\sin v) + \cos v\,(1) = \frac{dt}{dv}$$

$$(v\sin v - \cos v)\,dv = -dt$$

$$\therefore \quad -\log(v\cos v) = 2\log x + \log c$$

$$\log x^2 + \log(v\cos v) + \log c = 0$$

$$\log x^2 v\cos v = +\log c^{-1}$$

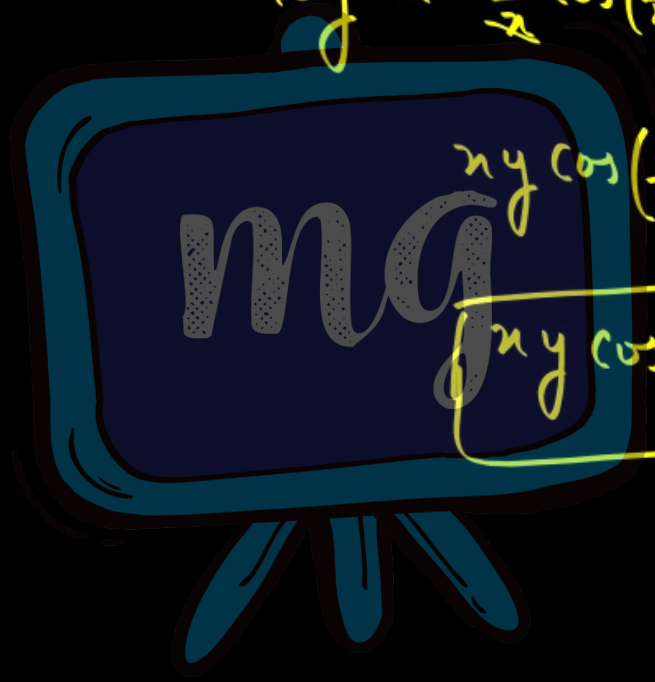
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$$\log x^n \cdot \frac{y}{x} \cos\left(\frac{y}{x}\right) = -\log c$$

$$x^n y \cos\left(\frac{y}{x}\right) = \frac{1}{c}$$

$$\boxed{x^n y \cos\left(\frac{y}{x}\right) = C'}$$



प्रश्नावली 9.4

11 से 15 तक के प्रश्नों में प्रत्येक अवकल समीकरण के लिए दिए हुए प्रतिबंध को संतुष्ट करने वाला विशिष्ट हल ज्ञात कीजिए।

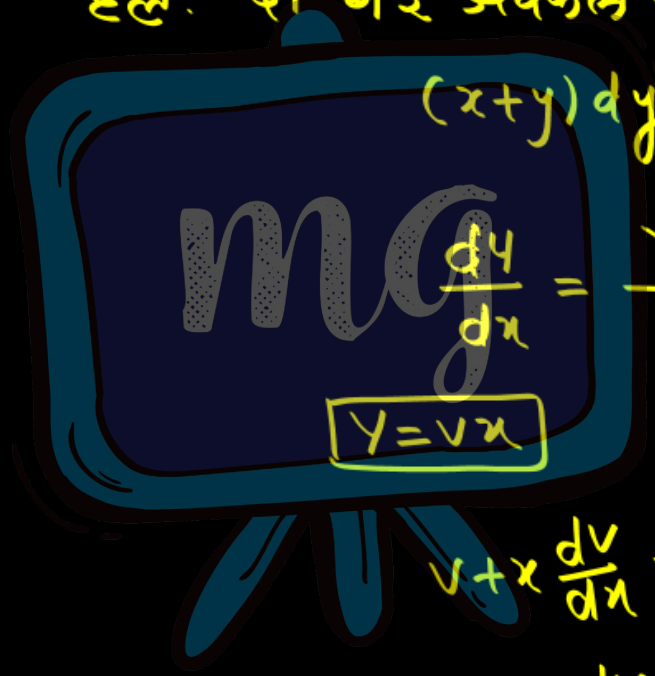
11. $(x + y)\, dy + (x - y)\, dx = 0;\ y = 1$ यदि $x = 1$

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हल: दी गई अवकल समी० –

$$(x+y)dy + (x-y)dx = 0$$



$$\frac{dy}{dx} = \frac{y-x}{y+x}$$

$y = vx$

$$v + x\frac{dv}{dx} = \frac{v-1}{v+1}$$

$$x\frac{dv}{dx} = \frac{v-1-v^2x}{v+1}$$

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$$\int \frac{v+1}{v^2+1}\,dv = \int -\frac{1}{x}\,dx$$

$$\frac{1}{2}\int \frac{2v}{v^2+1}\,dv + \int \frac{1}{v^2+1}\,dv = -\int \frac{1}{x}\,dx$$

$$\frac{1}{2}\log(v^2+1) + \tan^{-1}v = -\log x + C$$

$$\boxed{\frac{1}{2}\log\left|\frac{y^2}{x^2}+1\right| + \tan^{-1}\left(\frac{y}{x}\right) = -\log x + c}$$